

# Interpolation of Cesàro sequence and function spaces

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## Abstract

The interpolation property of Cesàro sequence and function spaces is investigated. It is shown that  $Ces_p(I)$  is an interpolation space between  $Ces_{p_0}(I)$  and  $Ces_{p_1}(I)$  for  $1 < p_0 < p_1 \leq \infty$  and  $1/p = (1 - \theta)/p_0 + \theta/p_1$  with  $0 < \theta < 1$ , where  $I = [0, \infty)$  or  $[0, 1]$ . The same result is true for Cesàro sequence spaces. On the other hand,  $Ces_p[0, 1]$  is not an interpolation space between  $Ces_1[0, 1]$  and  $Ces_\infty[0, 1]$ .

## 1. Introduction and preliminaries

Structure of the Cesàro sequence and function spaces was investigated by several authors (see, for example, [5], [15] and [2], [3] and references therein). Here we are interested in studying interpolation properties of Cesàro sequence and function spaces. The main purpose is to give interpolation theorems for the Cesàro sequence spaces  $ces_p$  and Cesàro function spaces  $Ces_p(I)$  on  $I = [0, \infty)$  and  $I = [0, 1]$ . In the case of  $I = [0, \infty)$  some interpolation results for Cesàro function spaces are contained implicitly in [16]. Moreover, using the so-called  $K^+$ -method of interpolation it was proved in [9] that Cesàro sequence space  $ces_p$  is an interpolation space with respect to the couple  $(l_1, l_1(2^{-k}))$ . Our main aim is to give a rather complete description of Cesàro spaces as interpolation spaces with respect to appropriate couples of weighted  $L_1$ -spaces as well as Cesàro spaces. For example, if either  $I = [0, \infty)$  or  $[0, 1]$  and  $1 < p_0 < p_1 \leq \infty$  with  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  for  $0 < \theta < 1$ , then

$$(Ces_{p_0}(I), Ces_{p_1}(I))_{\theta, p} = Ces_p(I) \quad \text{and} \quad (ces_{p_0}, ces_{p_1})_{\theta, p} = ces_p, \quad (1)$$

where  $(\cdot, \cdot)_{\theta, p}$  denotes the K-method of interpolation.

We have a completely different situation in a more interesting and non-trivial case when  $I = [0, 1]$  and  $p_0 = 1$ ,  $p_1 = \infty$ . It turns out that  $Ces_p[0, 1]$  is not an interpolation space between the spaces  $Ces_1[0, 1]$  and  $Ces_\infty[0, 1]$  and  $(Ces_1[0, 1], Ces_\infty[0, 1])_{\theta, p}$  for  $1 < p < \infty$  is a weighted Cesàro function space.

Let us collect some necessary definitions and notations related to the interpolation theory of operators as well as Cesàro, Copson and down spaces.

For two normed spaces  $X$  and  $Y$  the symbol  $X \xhookrightarrow{C} Y$  means that the imbedding  $X \subset Y$  is continuous with the norm which is not bigger than  $C$ , i.e.,  $\|x\|_Y \leq C\|x\|_X$  for

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all  $x \in X$ , and  $X \hookrightarrow Y$  means that  $X \xhookrightarrow{C} Y$  for some  $C > 0$ . Moreover,  $X = Y$  means that  $X \hookrightarrow Y$  and  $Y \hookrightarrow X$ , that is, the spaces are the same and the norms are equivalent. If  $f$  and  $g$  are real functions, then the symbol  $f \approx g$  means that  $c^{-1}g \leq f \leq cg$  for some  $c \geq 1$ .

For a Banach couple  $\bar{X} = (X_0, X_1)$  of two compatible Banach spaces  $X_0$  and  $X_1$  consider two Banach spaces  $X_0 \cap X_1$  and  $X_0 + X_1$  with its natural norms

$$\|f\|_{X_0 \cap X_1} = \max(\|f\|_{X_0}, \|f\|_{X_1}) \quad \text{for } f \in X_0 \cap X_1,$$

and

$$\|f\|_{X_0 + X_1} = \inf \{\|f_0\|_{X_0} + \|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\} \quad \text{for } f \in X_0 + X_1.$$

For more careful definitions of a Banach couple, intermediate and interpolation spaces with some results introduced briefly below, see [8, pp. 91-173, 289-314, 338-359] and [6, pp. 95-116].

A Banach space  $X$  is called an *intermediate space* between  $X_0$  and  $X_1$  if  $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$ . Such a space  $X$  is called an *interpolation space* between  $X_0$  and  $X_1$  (we write:  $X \in \text{Int}(X_0, X_1)$ ) if, for any bounded linear operator  $T : X_0 + X_1 \rightarrow X_0 + X_1$  such that the restriction  $T|_{X_i} : X_i \rightarrow X_i$  is bounded for  $i = 0, 1$ , the restriction  $T|_X : X \rightarrow X$  is also bounded and  $\|T\|_{X \rightarrow X} \leq C \max \{\|T\|_{X_0 \rightarrow X_0}, \|T\|_{X_1 \rightarrow X_1}\}$  for some  $C \geq 1$ . If  $C = 1$ , then  $X$  is called an *exact interpolation space* between  $X_0$  and  $X_1$ .

An *interpolation method* or *interpolation functor*  $\mathcal{F}$  is a construction (a rule) which assigns to every Banach couple  $\bar{X} = (X_0, X_1)$  an interpolation space  $\mathcal{F}(\bar{X})$  between  $X_0$  and  $X_1$ . The interpolation functor  $\mathcal{F}$  is called *exact* if the space  $\mathcal{F}(\bar{X})$  is an exact interpolation space for every Banach couple  $\bar{X}$ . One of the most important interpolation methods is the *K-method* known also as the *real Lions-Peetre interpolation method*. For a Banach couple  $\bar{X} = (X_0, X_1)$  the *Peetre K-functional* of an element  $f \in X_0 + X_1$  is defined for  $t > 0$  by

$$K(t, f; X_0, X_1) = \inf \{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\}.$$

Then the *spaces of the K-method of interpolation* are

$$(X_0, X_1)_{\theta, p} = \{f \in X_0 + X_1 : \|f\|_{\theta, p} = \left( \int_0^\infty [t^{-\theta} K(t, f; X_0, X_1)]^p \frac{dt}{t} \right)^{1/p} < \infty\}$$

if  $0 < \theta < 1$  and  $1 \leq p < \infty$ , and

$$(X_0, X_1)_{\theta, \infty} = \{f \in X_0 + X_1 : \|f\|_{\theta, \infty} = \sup_{t>0} \frac{K(t, f; X_0, X_1)}{t^\theta} < \infty\}$$

if  $0 \leq \theta \leq 1$ . Very useful in calculations is the so-called *reiteration formula* showing the stability of the *K-method* of interpolation. If  $1 \leq p_0, p_1, p \leq \infty, 0 < \theta_0, \theta_1, \theta < 1$  and  $\theta_0 \neq \theta_1$ , then with equivalent norms

$$((X_0, X_1)_{\theta_0, p_0}, (X_0, X_1)_{\theta_1, p_1})_{\theta, p} = (X_0, X_1)_{\eta, p}, \quad (2)$$

where  $\eta = (1 - \theta)\theta_0 + \theta\theta_1$  (see [6, Theorem 2.4, p. 311], [7, Theorems 3.5.3], [8, Theorem 3.8.10]) and [23, Theorem 1.10.2]).

The space  $(X_0, X_1)_{\Phi}^K$  of the *general K-method of interpolation*, where  $\Phi$  is a *parameter of the K-method*, i.e., a Banach function space on  $((0, \infty), dt/t)$  containing the function  $t \mapsto \min\{1, t\}$ , is the Banach space of all  $f \in X_0 + X_1$  such that  $K(\cdot, f; X_0, X_1) \in \Phi$  with the norm  $\|f\|_{K_{\Phi}} = \|K(\cdot, f; X_0, X_1)\|_{\Phi}$ . The space  $(X_0, X_1)_{\Phi}^K$  is an exact interpolation space between  $X_0$  and  $X_1$ .

In particular, if  $L_p = L_p(\Omega, \mu)$ , where  $(\Omega, \mu)$  is a complete  $\sigma$ -finite measure space, then for any  $f \in L_1 + L_{\infty}$  we have

$$K(t, f; L_1, L_{\infty}) = \int_0^t f^*(s) ds. \quad (3)$$

Here and next  $f^*$  denotes the non-increasing rearrangement of  $|f|$  defined by  $f^*(s) = \inf\{\lambda > 0 : \mu(\{x \in \Omega : |f(x)| > \lambda\}) \leq s\}$  (see [8, Proposition 3.1.1], [12, pp. 78-79], [6, Theorem 6.2, pp. 74-75]). Moreover, for two non-negative weight functions  $w_0, w_1$  and for  $f \in L_1(w_0) + L_1(w_1)$  we have

$$K(t, f; L_1(w_0), L_1(w_1)) = \|\min(w_0, tw_1) f\|_{L_1} \quad (4)$$

(see [8, Proposition 3.1.17] and [18, p. 391]).

If the inequality  $K(t, g; X_0, X_1) \leq K(t, f; X_0, X_1)$  ( $t > 0$ ) with  $f \in X$  and  $g \in X_0 + X_1$  implies that  $g \in X$  and  $\|g\|_X \leq C \|f\|_X$  for arbitrary  $X \in \text{Int}(X_0, X_1)$  and some  $C \geq 1$  independent of  $X, f$  and  $g$ , then  $(X_0, X_1)$  is called a *K-monotone* or *Calderón-Mityagin couple*. For arbitrary *K-monotone couple*  $(X_0, X_1)$  the spaces  $(X_0, X_1)_{\Phi}^K$  of the general *K-method* are the only interpolation spaces between  $X_0$  and  $X_1$  (see [8]).

Now, to treat interpolation results for Cesàro spaces we need to define these spaces. The *Cesàro sequence spaces*  $ces_p$  are the sets of real sequences  $x = \{x_k\}$  such that

$$\|x\|_{ces(p)} = \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=1}^n |x_k| \right)^p \right]^{1/p} < \infty, \quad \text{for } 1 \leq p < \infty,$$

and

$$\|x\|_{ces(\infty)} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^n |x_k| < \infty, \quad \text{for } p = \infty.$$

The *Cesàro function spaces*  $Ces_p = Ces_p(I)$  are the classes of Lebesgue measurable real functions  $f$  on  $I = [0, 1]$  or  $I = [0, \infty)$  such that

$$\|f\|_{Ces(p)} = \left[ \int_I \left( \frac{1}{x} \int_0^x |f(t)| dt \right)^p dx \right]^{1/p} < \infty, \quad \text{for } 1 \leq p < \infty,$$

and

$$\|f\|_{Ces(\infty)} = \sup_{0 < x \in I} \frac{1}{x} \int_0^x |f(t)| dt < \infty, \quad \text{for } p = \infty.$$

The Cesàro spaces are Banach lattices which are not symmetric except as they are trivial, namely,  $ces_1 = \{0\}$ ,  $Ces_1[0, \infty) = \{0\}$ . By a *symmetric space* we mean a Banach lattice  $X$  on  $I$  satisfying the additional property: if  $g^*(t) = f^*(t)$  for all  $t > 0$ ,  $f \in X$  and  $g \in L^0(I)$  (the set of all classes of Lebesgue measurable real functions on  $I$ ) then  $g \in X$

and  $\|g\|_X = \|f\|_X$  (cf. [6], [12]). Moreover,  $l_p \xrightarrow{p'} ces_p, L_p(I) \xrightarrow{p'} Ces_p(I)$  for  $1 < p \leq \infty$  (in what follows  $\frac{1}{p} + \frac{1}{p'} = 1$ ), and if  $1 < p < q < \infty$ , then  $ces_p \xrightarrow{1} ces_q \xrightarrow{1} ces_\infty$ . Also for  $I = [0, 1]$  and  $1 < p < q < \infty$  we have  $L_\infty \xrightarrow{1} Ces_\infty \xrightarrow{1} Ces_q \xrightarrow{1} Ces_p \xrightarrow{1} Ces_1 = L_1(\ln 1/t)$  and  $Ces_\infty \xrightarrow{1} L_1$ .

Let  $1 \leq p < \infty$ . The *Copson sequence spaces*  $cop_p$  are the sets of real sequences  $x = \{x_k\}$  such that

$$\|x\|_{cop(p)} = \left[ \sum_{n=1}^{\infty} \left( \sum_{k=n}^{\infty} \frac{|x_k|}{k} \right)^p \right]^{1/p} < \infty,$$

and the *Copson function spaces*  $Cop_p = Cop_p(I)$  are the classes of Lebesgue measurable real functions  $f$  on  $I = [0, \infty)$  or  $I = [0, 1]$  such that

$$\|f\|_{Cop(p)} = \left[ \int_0^\infty \left( \int_x^\infty \frac{|f(t)|}{t} dt \right)^p dx \right]^{1/p} < \infty \quad \text{for } I = [0, \infty),$$

and

$$\|f\|_{Cop(p)} = \left[ \int_0^1 \left( \int_x^1 \frac{|f(t)|}{t} dt \right)^p dx \right]^{1/p} < \infty \quad \text{for } I = [0, 1].$$

Sometimes we will use the *Cesàro operators*  $C_d x(n) = \frac{1}{n} \sum_{k=1}^n |x_k|$ ,  $Cf(x) = \frac{1}{x} \int_0^x |f(t)| dt$  and the *Copson operators*  $C_d^* x(n) = \sum_{k=n}^\infty \frac{|x_k|}{k}$ ,  $C^* f(x) = \int_{(x, \infty) \cap I} \frac{|f(t)|}{t} dt$  related to introduced spaces. Then  $ces_p$  (resp.  $cop_p$ ) consists of all real sequences  $x = \{x_k\}$  such that  $C_d x \in l_p$  (resp.  $C_d^* x \in l_p$ ) and  $Ces_p(I)$  (resp.  $Cop_p(I)$ ) consists of all classes of Lebesgue measurable real functions  $f$  on  $I$  such that  $Cf \in L_p(I)$  (resp.  $C^* f \in L_p(I)$ ) with natural norms. By the Copson inequalities (cf. [11, Theorems 328 and 331], [5, p. 25] and [13, p. 159]), which are valid for  $1 \leq p < \infty$ , we have:  $\|C_d^* x\|_{l_p} \leq p \|x\|_{l_p}$  for  $x \in l_p$  and  $\|C^* f\|_{L_p(I)} \leq p \|f\|_{L_p(I)}$  for  $f \in L_p(I)$ . Therefore,  $l_p \xrightarrow{p} cop_p, L_p \xrightarrow{p} Cop_p$ .

We can define similarly the spaces  $cop_\infty$  and  $Cop_\infty$  but then it is easy to see that  $cop_\infty = l_1(1/k)$  and  $Cop_\infty = L_1(1/t)$ . Moreover, for  $I = [0, 1]$  we have  $L_p \xrightarrow{p} Cop_p \xrightarrow{1} Cop_1 = L_1$ .

We will consider also more general Cesàro spaces  $Ces_E(I)$ , where  $E$  is a Banach function space on  $I$  with the natural norm  $\|f\|_{Ces(E)} = \|Cf\|_E$ .

For a Banach function space  $E$  on  $I = [0, \infty)$  the *down space*  $E^\downarrow$  is the collection of all  $f \in L^0$  such that the norm

$$\|f\|_{E^\downarrow} = \sup \int_I |f(t)|g(t) dt < \infty,$$

where the supremum is taken over all non-negative, non-increasing Lebesgue measurable functions  $g$  from the Köthe dual  $E'$  of  $E$  such that  $\|g\|_{E'} \leq 1$ . Let us remind that the Köthe dual of a Banach function space  $E$  is defined as

$$E' = \{f \in L^0 : \|f\|_{E'} = \sup_{\|g\|_E \leq 1} \int_I |f(t)g(t)| dt < \infty\}.$$

It is routine to check that the space  $E^\downarrow$  has the Fatou property, that is, if  $0 \leq f_n$  increases to  $f$  a.e. on  $I$  and  $\sup_{n \in \mathbb{N}} \|f_n\|_{E^\downarrow} < \infty$ , then  $f \in E^\downarrow$  and  $\|f_n\|_{E^\downarrow}$  increases to  $\|f\|_{E^\downarrow}$ .

Moreover,  $E'' \xrightarrow{1} E^\downarrow$ , where  $E''$  is the second Köthe dual of  $E$ . Recall also that a Banach function space  $E$  has the Fatou property if and only if  $E = E''$  with the equality of the norms.

Sinnamon ([21], Theorem 3.1) proved that if  $E$  is a symmetric space on  $I = [0, \infty)$ , then  $\|f\|_{E^\downarrow} \approx \|Cf\|_E$  if and only if the Cesàro operator  $C : E \rightarrow E$  is bounded. In particular, then  $E^\downarrow = Ces_E$ . Moreover,  $(L_1)^\downarrow = L_1$  since

$$\|f\|_{L_1^\downarrow} = \sup_{0 \leq g} \frac{\int_0^\infty |f(t)|g(t) dt}{\|g\|_{L_\infty}} \geq \sup_{0 \leq g^\downarrow} \frac{\int_0^\infty |f(t)|g(t) dt}{\|g\|_{L_\infty}} \geq \frac{\int_0^\infty |f(t)| dt}{\|1\|_{L_\infty}} = \|f\|_{L_1}$$

(cf. [16], p. 194).

The paper is organized as follows. In Section 2 we proved that the Cesàro and Copson sequence and function spaces on  $[0, \infty)$  are interpolation spaces obtained by the  $K$ -method from weighted  $L_1$ -spaces. At the same time, in the case of  $I = [0, 1]$ , the  $K$ -method gives only the Copson spaces as interpolation spaces with respect to analogous couple of weighted  $L_1$ -spaces (see Theorem 1(iii)). In particular, we obtain a new description of the interpolation spaces  $(L_1, L_1(1/t))_{1-1/p, p}$  in off-diagonal case both for  $I = [0, \infty)$  and  $I = [0, 1]$ .

In Section 3 it is shown that the Cesàro function spaces  $Ces_p[0, \infty)$ ,  $1 < p < \infty$  can be obtained by the  $K$ -method of interpolation also from the couple  $(L_1[0, \infty), Ces_\infty[0, \infty))$ . Hence, applying the reiteration theorem, we conclude that  $Ces_p[0, \infty)$  are interpolation spaces with respect to the couple  $(Ces_{p_0}[0, \infty), Ces_{p_1}[0, \infty))$  for arbitrary  $1 < p_0 < p_1 \leq \infty$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  with  $0 < \theta < 1$ .

In Section 4 the interpolation of Cesàro function spaces on the segment  $[0, 1]$  is investigated. We prove that for  $1 < p < \infty$

$$(L_1(1-t)[0, 1], Ces_\infty[0, 1])_{\theta, p} = Ces_p[0, 1] \quad \text{with } \theta = 1 - 1/p.$$

As a consequence of this result and reiteration equality (2), we infer

$$(Ces_{p_0}[0, 1], Ces_{p_1}[0, 1])_{\theta, p} = Ces_p[0, 1] \tag{5}$$

for all  $1 < p_0 < p_1 \leq \infty$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  with  $0 < \theta < 1$ .

Our important part of interest is to look on interpolation spaces between the spaces  $Ces_1[0, 1]$  and  $Ces_\infty[0, 1]$ . In Section 5, in Theorem 3, we find an equivalent expression for the  $K$ -functional with respect to this couple and then in Section 6 we proved that the real interpolation spaces  $(Ces_1[0, 1], Ces_\infty[0, 1])_{1-1/p, p}$  for  $1 < p < \infty$  can be identified with the weighted Cesàro function spaces  $Ces_p(\ln e/t)[0, 1]$ .

Finally, in Section 7, we show in Theorem 6 that  $Ces_p[0, 1]$  for  $1 < p < \infty$  are not interpolation spaces between the spaces  $Ces_1[0, 1]$  and  $Ces_\infty[0, 1]$ .

## 2. Cesàro and Copson spaces as interpolation spaces with respect to weighted $L_1$ -spaces

We start with the main result in this part.

**THEOREM 1.** (i) If  $1 < p < \infty$ , then

$$(l_1, l_1(1/k))_{1-1/p, p} = ces_p = cop_p.$$

(ii) If  $I = [0, \infty)$  and  $1 < p < \infty$ , then

$$(L_1, L_1(1/t))_{1-1/p, p} = Ces_p = Cop_p.$$

(iii) If  $I = [0, 1]$  and  $1 < p \leq \infty$ , then

$$(L_1, L_1(1/t))_{1-1/p, p} = Cop_p.$$

Moreover,  $Cop_p \xrightarrow{p'} Ces_p$  and the reverse imbedding does not hold.

*Proof.* (i) If  $f \in l_1 + l_1(1/k)$ , then  $K(t, x; l_1, l_1(1/k)) = t \sum_{k=1}^{\infty} \frac{|x_k|}{k}$  for  $0 < t \leq 1$ , and

$$K(t, x; l_1, l_1(1/k)) = \sum_{k=1}^{\infty} |x_k| \min(1, \frac{t}{k}) = \sum_{k=1}^{[t]} |x_k| + t \sum_{k=[t]+1}^{\infty} \frac{|x_k|}{k}.$$

for  $t \geq 1$ . Therefore, for  $n \leq t < n+1$  ( $n \geq 1$ ), we have

$$\frac{K(t, x; l_1, l_1(1/k))}{t} \leq \frac{1}{n} \sum_{k=1}^n |x_k| + \sum_{k=n+1}^{\infty} \frac{|x_k|}{k} = C_d x(n) + C_d^* x(n+1).$$

Since

$$\begin{aligned} C_d C_d^* x(n) &= \frac{1}{n} \sum_{m=1}^n \left( \sum_{k=m}^{\infty} \frac{|x_k|}{k} \right) = \frac{1}{n} \left[ \sum_{k=1}^n \left( \sum_{m=1}^k \frac{|x_k|}{k} \right) + \sum_{k=n+1}^{\infty} \left( \sum_{m=1}^n \frac{|x_k|}{k} \right) \right] \\ &= \frac{1}{n} \sum_{k=1}^n |x_k| + \sum_{k=n+1}^{\infty} \frac{|x_k|}{k} = C_d x(n) + C_d^* x(n+1), \end{aligned}$$

it follows that, for  $n \leq t < n+1$  ( $n \geq 1$ ),

$$\frac{K(t, x; l_1, l_1(1/k))}{t} \leq C_d C_d^* x(n).$$

Using the classical Hardy inequality (cf. [11, Theorem 326] or [13, Theorem 1]), we obtain

$$\begin{aligned} \|x\|_{1-1/p, p} &= \left( \int_0^{\infty} \left( \frac{K(t, x; l_1, l_1(1/k))}{t} \right)^p dt \right)^{1/p} \\ &= \left[ C_d^* x(1)^p + \sum_{n=1}^{\infty} \int_n^{n+1} \left( \frac{K(t, x)}{t} \right)^p dt \right]^{1/p} \\ &\leq \left[ C_d^* x(1)^p + \sum_{n=1}^{\infty} (C_d C_d^* x(n))^p \right]^{1/p} \\ &\leq C_d^* x(1) + \|C_d C_d^* x\|_{l_p} \leq C_d^* x(1) + p' \|C_d^* x\|_{l_p} \\ &\leq (p' + 1) \|C_d^* x\|_{l_p} = (p' + 1) \|x\|_{cop(p)}. \end{aligned}$$

This means that  $\text{cop}_p \hookrightarrow (l_1, l_1(1/k))_{1-1/p, p}$ . On the other hand, for  $n \leq t < n+1$  ( $n \geq 1$ ), we have

$$\frac{K(t, x; l_1, l_1(1/k))}{t} \geq \sum_{k=n+1}^{\infty} \frac{|x_k|}{k} = C_d^* x(n+1)$$

and

$$\begin{aligned} \|x\|_{1-1/p, p} &= \left( \int_0^\infty \left( \frac{K(t, x; l_1, l_1(1/k))}{t} \right)^p dt \right)^{1/p} \\ &\geq \left( C_d^* x(1)^p + \sum_{n=1}^{\infty} C_d^* x(n+1)^p \right)^{1/p} = \|C_d^* x\|_{l_p} = \|x\|_{\text{cop}(p)}, \end{aligned}$$

which gives the reverse imbedding  $(l_1, l_1(1/k))_{1-1/p, p} \xrightarrow{1} \text{cop}_p$ . The equality of the spaces  $\text{ces}_p = \text{cop}_p$  for  $1 < p < \infty$  was proved by Bennett (cf. [5], Theorems 4.5 and 6.6).

(ii) For  $f \in L_1 + L_1(1/s) = L_1(\min(1, 1/s))$  we have

$$K(t, f; L_1, L_1(1/s)) = \int_0^\infty |f(s)| \min(1, t/s) ds = \int_0^t |f(s)| ds + t \int_t^\infty \frac{|f(s)|}{s} ds.$$

Thus,

$$\frac{K(t, f; L_1, L_1(1/s))}{t} = Cf(t) + C^*f(t), \quad t > 0,$$

and therefore

$$\|f\|_{1-1/p, p} = \left( \int_0^\infty \left( \frac{K(t, f; L_1, L_1(1/s))}{t} \right)^p dt \right)^{1/p} = \|Cf + C^*f\|_{L_p(0, \infty)}. \quad (6)$$

Since, by Fubini theorem,

$$\begin{aligned} C^*Cf(t) &= \int_t^\infty \left( \frac{1}{u^2} \int_0^u |f(s)| ds \right) du \\ &= \int_0^t \left( \int_t^\infty \frac{1}{u^2} du \right) |f(s)| ds + \int_t^\infty \left( \int_s^\infty \frac{1}{u^2} du \right) |f(s)| ds \\ &= \frac{1}{t} \int_0^t |f(s)| ds + \int_t^\infty \frac{|f(s)|}{s} ds = Cf(t) + C^*f(t), \end{aligned}$$

from the Copson inequality (cf. [11, Theorem 328]) it follows that

$$\begin{aligned} \|f\|_{\text{ces}(p)} &= \|Cf\|_{L_p(0, \infty)} \leq \|Cf + C^*f\|_{L_p(0, \infty)} \\ &= \|C^*Cf\|_{L_p(0, \infty)} \leq p \|Cf\|_{L_p(0, \infty)} = p \|f\|_{\text{ces}(p)}. \end{aligned}$$

Combining this with (6), we obtain  $\|f\|_{1-1/p, p} \approx \|f\|_{\text{ces}(p)}$ .

On the other hand, since

$$\begin{aligned} CC^*f(t) &= \frac{1}{t} \int_0^t \left( \int_u^\infty \frac{|f(s)|}{s} ds \right) du \\ &= \frac{1}{t} \int_0^t \left( \int_0^s du \right) \frac{|f(s)|}{s} ds + \frac{1}{t} \int_t^\infty \left( \int_0^t du \right) \frac{|f(s)|}{s} ds \\ &= \frac{1}{t} \int_0^t |f(s)| ds + \int_t^\infty \frac{|f(s)|}{s} ds = Cf(t) + C^*f(t), \end{aligned}$$

then, by Hardy inequality,

$$\begin{aligned}\|f\|_{Cop(p)} &= \|C^*f\|_{L_p(0,\infty)} \leq \|Cf + C^*f\|_{L_p(0,\infty)} \\ &= \|CC^*f\|_{L_p(0,\infty)} \leq p'\|C^*f\|_{L_p(0,\infty)} = p'\|f\|_{Cop(p)},\end{aligned}$$

and, applying (6) once more, we conclude that  $\|f\|_{1-1/p,p} \approx \|f\|_{Cop(p)}$ .

(iii) For  $I = [0, 1]$  and  $f \in L_1 + L_1(1/s) = L_1$  we have  $K(t, f; L_1, L_1(1/s)) = \|f\|_1$  for  $t \geq 1$  and

$$K(t, f; L_1, L_1(1/s)) = \int_0^t |f(s)| ds + t \int_t^1 \frac{|f(s)|}{s} ds = tCf(t) + tC^*f(t)$$

for  $0 < t \leq 1$ . Therefore, for  $1 < p < \infty$

$$\begin{aligned}\|f\|_{1-1/p,p} &= \left( \int_0^1 [Cf(t) + C^*f(t)]^p dt + \int_1^\infty t^{-p} \|f\|_1^p dt \right)^{1/p} \\ &= \left( \|Cf + C^*f\|_p^p + \frac{1}{p-1} \|f\|_1^p \right)^{1/p}.\end{aligned}$$

Firstly, the last expression is not smaller than  $\|C^*f\|_p = \|f\|_{Cop(p)}$ . On the other hand, since again  $CC^*f(t) = Cf(t) + C^*f(t)$ , by Hardy inequality, it follows that

$$\begin{aligned}\|f\|_{1-1/p,p} &= \left( \|CC^*f\|_p^p + \frac{1}{p-1} \|f\|_1^p \right)^{1/p} \leq \|CC^*f\|_p + (p-1)^{-1/p} \|f\|_1 \\ &\leq p'\|C^*f\|_p + (p-1)^{-1/p} \|f\|_{Cop(p)} = (p' + (p-1)^{-1/p}) \|f\|_{Cop(p)}.\end{aligned}$$

Thus,  $(L_1, L_1(1/t))_{1-1/p,p} = Cop_p$  with equivalent norms for  $1 < p < \infty$ . For  $p = \infty$  we have  $(L_1, L_1(1/t))_{1,\infty} = L_1(1/t) = Cop_\infty[0, 1]$ .

The imbedding  $Cop_p \xrightarrow{p'} Ces_p$  for  $1 < p \leq \infty$  follows from the inequality

$$\|f\|_{Ces(p)} = \|Cf\|_p \leq \|Cf + C^*f\|_p = \|CC^*f\|_p \leq p'\|C^*f\|_p = p'\|f\|_{Cop(p)}.$$

Moreover,  $Ces_p[0, 1] \cap L_1[0, 1] \xrightarrow{p+1} Cop_p[0, 1]$  for  $1 \leq p < \infty$ . In fact, observe that in the case of  $I = [0, 1]$  the composition operator  $C^*C$  has an additional term. More precisely,

$$C^*Cf(t) = Cf(t) + C^*f(t) - \int_0^1 |f(s)| ds.$$

Therefore,

$$\begin{aligned}\|f\|_{Cop(p)} &= \|C^*f\|_p \leq \|Cf + C^*f\|_p \\ &= \|C^*Cf + \int_0^1 |f(s)| ds\|_p \leq \|C^*Cf\|_p + \|f\|_1 \\ &\leq p\|Cf\|_p + \|f\|_1 \leq (p+1) \max(\|f\|_{Ces(p)}, \|f\|_1).\end{aligned}$$



Finally, let us show that  $Ces_p \not\hookrightarrow Cop_p$  by comparing norms of the functions  $f_h(t) = \frac{1}{\sqrt{1-t}}\chi_{[h,1)}(t)$ ,  $0 < h < 1$  in these spaces. We have

$$C^*(f_h)(t) = \begin{cases} \int_h^1 \frac{1}{s\sqrt{1-s}} ds, & \text{if } 0 < t \leq h, \\ \int_t^1 \frac{1}{s\sqrt{1-s}} ds, & \text{if } h \leq t \leq 1, \end{cases}$$

and

$$\begin{aligned} \|f_h\|_{Cop(p)}^p &= \|C^*(f_h)\|_p^p \geq \int_0^h \left( \int_h^1 \frac{1}{s\sqrt{1-s}} ds \right)^p dt \\ &= h \left( \int_h^1 \frac{1}{s\sqrt{1-s}} ds \right)^p \geq h \left( \int_h^1 \frac{1}{\sqrt{1-s}} ds \right)^p \\ &= 2^p h (1-h)^{p/2}. \end{aligned}$$

Also,

$$C(f_h)(t) = \begin{cases} 0, & \text{if } 0 < t \leq h, \\ \frac{2}{t}(\sqrt{1-h} - \sqrt{1-t}), & \text{if } h \leq t \leq 1, \end{cases}$$

and

$$\begin{aligned} \|f_h\|_{Ces(p)}^p &= \|C(f_h)\|_p^p = 2^p \int_h^1 \left( \frac{\sqrt{1-h} - \sqrt{1-t}}{t} \right)^p dt \\ &\leq 2^p \int_h^1 \frac{(1-h)^{p/2}}{t^p} dt = 2^p (1-h)^{p/2} \frac{1-h^{p-1}}{(p-1)h^{p-1}}. \end{aligned}$$

Thus

$$\frac{\|f_h\|_{Cop(p)}^p}{\|f_h\|_{Ces(p)}^p} \geq \frac{2^p h (1-h)^{p/2} (p-1) h^{p-1}}{2^p (1-h)^{p/2} (1-h^{p-1})} = (p-1) \frac{h^p}{1-h^{p-1}} \rightarrow \infty \text{ as } h \rightarrow 1^+,$$

and the proof is complete.  $\square$

**Remark 1.** Alternatively, the space  $ces_p$  for  $1 < p < \infty$  can be obtained as an interpolation space with respect to the couple  $(l_1, l_1(2^{-n}))$  by the so-called  $K^+$ -method being a version of the standard K-method, precisely,  $ces_p = (l_1, l_1(2^{-n}))_{l_p(1/n)}^{K^+}$  (cf. [9, the proof of Theorem 6.4]).

**Remark 2.** The results in Theorem 1 give a description of the real interpolation spaces  $(L_1, L_1(1/t))_{1-1/p, p}$  in the off-diagonal case. Before it was only known that they are intersections of weighted  $L_1(w)$ -spaces with the weights  $w$  from certain sets (cf. [10, Theorem 4.1], [14, Theorem 2]) or some block spaces (cf. [1, Lemma 3.1]).

The following corollary follows directly from Theorem 1, reiteration formula (2) and the equalities  $Cop_\infty[0, 1] = L_1(1/t)$  and  $Cop_1[0, 1] = L_1$ .

**Corollary 1.** *If  $1 < p_0 < p_1 < \infty$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  with  $0 < \theta < 1$ , then*

$$(ces_{p_0}, ces_{p_1})_{\theta, p} = ces_p, (Ces_{p_0}[0, \infty), Ces_{p_1}[0, \infty))_{\theta, p} = Ces_p[0, \infty). \quad (7)$$

*If  $1 \leq p_0 < p_1 \leq \infty$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  with  $0 < \theta < 1$ , then*

$$(Cop_{p_0}[0, 1], Cop_{p_1}[0, 1])_{\theta, p} = Cop_p[0, 1]. \quad (8)$$

**Remark 3.** Another proof of the second equality for the spaces on  $[0, \infty)$  from the last corollary was given by Sinnamon [19, Corollary 2].

### 3. Cesàro spaces on $[0, \infty)$ as interpolation spaces with respect to the couple $(L_1, Ces_\infty)$

All the spaces considered in this part are on the interval  $I = [0, \infty)$ . By [16, p. 194] the down space  $D^\infty := (L_\infty)^\downarrow = Ces_\infty$  isometrically. On the other hand, for a Banach lattice  $F$  with the Fatou property we have  $F \in Int(L_1, D^\infty) = Int(L_1, Ces_\infty)$  if and only if  $F = E^\downarrow$  with equality of norms for some  $E \in Int(L_1, L_\infty)$  (see [16, Theorem 6.4]). Then, in particular,  $L_p^\downarrow \in Int(L_1, Ces_\infty)$ . Since the operator  $C$  is bounded in  $L_p$  for  $1 < p \leq \infty$ , by [21, Theorem 3.1], it follows that

$$\|f\|_{L_p^\downarrow} = \| |f| \|_{L_p^\downarrow} \approx \|Cf\|_{L_p} = \|f\|_{Ces(p)}.$$

Thus, for any  $1 < p < \infty$  we have  $Ces_p \in Int(L_1, Ces_\infty)$  and  $Ces_p = L_p^\downarrow$ . Moreover, the following more precise and general assertion, which is an almost immediate consequence of Theorem 6.4 from [16], holds.

**Proposition 1.** *Let  $E, F \in Int(L_1, L_\infty)$  and  $\Phi$  be an interpolation Banach lattice with respect to the couple  $(L_\infty, L_\infty(1/u))$  on  $(0, \infty)$ . Then we have*

$$(E^\downarrow, F^\downarrow)_\Phi^K = [(E, F)_\Phi^K]^\downarrow. \quad (9)$$

*In particular, if  $1 < p < \infty$ , then*

$$(L_1, Ces_\infty)_{1-1/p, p} = Ces_p. \quad (10)$$

*Proof.* Firstly, since the Banach couple  $(L_1, L_\infty)$  is  $K$ -monotone [12, Theorem 2.4.3], by the assumption and the Brudnyi-Krugljak theorem (cf. [8, Theorem 4.4.5]),  $E = (L_1, L_\infty)_{\Phi_0}^K$  and  $F = (L_1, L_\infty)_{\Phi_1}^K$  with some interpolation Banach lattices  $\Phi_0$  and  $\Phi_1$  with respect to the couple  $(L_\infty, L_\infty(1/u))$  on  $(0, \infty)$ . Applying the reiteration theorem for the general  $K$ -method (see [8, Theorem 3.3.11]), we obtain

$$(E, F)_\Phi^K = ((L_1, L_\infty)_{\Phi_0}^K, (L_1, L_\infty)_{\Phi_1}^K)_\Phi^K = (L_1, L_\infty)_\Psi^K,$$

where  $\Psi = (\Phi_0, \Phi_1)_\Phi^K$ . Moreover, from the proof of Theorem 6.4 in [16] and the equality  $L_1^\downarrow = L_1$  (see Section 1) it follows

$$E^\downarrow = [(L_1, L_\infty)_{\Phi_0}^K]^\downarrow = (L_1, D^\infty)_{\Phi_0}^K, \quad F^\downarrow = [(L_1, L_\infty)_{\Phi_1}^K]^\downarrow = (L_1, D^\infty)_{\Phi_1}^K$$

and

$$[(E, F)_\Phi^K]^\downarrow = [(L_1, L_\infty)_\Psi^K]^\downarrow = (L_1, D^\infty)_\Psi^K.$$

Therefore, using the reiteration theorem once again, we obtain

$$(E^\downarrow, F^\downarrow)_\Phi^K = ((L_1, D^\infty)_{\Phi_0}^K, (L_1, D^\infty)_{\Phi_1}^K)_\Phi^K = (L_1, D^\infty)_\Psi^K = [(E, F)_\Phi^K]^\downarrow.$$

and equality (9) is proved. In particular, from (9) and the well-known identification formula  $(L_1, L_\infty)_{1-1/p, p} = L_p$  [7, Theorem 5.2.1] it follows that

$$(L_1, Ces_\infty)_{1-1/p, p} = (L_1^\downarrow, L_\infty^\downarrow)_{1-1/p, p} = L_p^\downarrow = Ces_p.$$

and also equality (10) is proved.  $\square$

For a given symmetric space  $E$  on  $I = [0, \infty)$  the Cesàro function space  $Ces_E$  is defined by the norm  $\|f\|_{Ces(E)} = \|Cf\|_E$ . If operator  $C$  is bounded in  $E$ , then, by [21, Theorem 3.1],  $Ces_E = E^\downarrow$ . Therefore, applying Proposition 1, we obtain

**Corollary 2.** *Let the operator  $C$  be bounded in symmetric spaces  $E$  and  $F$  on  $[0, \infty)$  and let  $\Phi$  be an interpolation Banach lattice with respect to the couple  $(L_\infty, L_\infty(1/u))$  on  $(0, \infty)$ . Then*

$$(Ces_E, Ces_F)_\Phi^K = Ces_{(E,F)_\Phi^K}.$$

In particular, for arbitrary  $1 < p_0 < p_1 \leq \infty$

$$(Ces_{p_0}, Ces_{p_1})_{\theta,p} = Ces_p, \text{ where } 0 < \theta < 1 \text{ and } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \quad (11)$$

**Remark 4.** If  $1 < p < \infty$ , then the restriction of the space  $Ces_p[0, \infty)$  to the interval  $[0, 1]$  coincides with the intersection  $Ces_p[0, 1] \cap L_1[0, 1]$  (cf. [3], Remark 5). Therefore, if we “restrict” formula (11) to  $[0, 1]$  we obtain only

$$(Ces_{p_0}[0, 1] \cap L_1[0, 1], Ces_{p_1}[0, 1] \cap L_1[0, 1])_{\theta,p} = Ces_p[0, 1] \cap L_1[0, 1],$$

where  $1 < p_0 < p_1 < \infty$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ .

#### 4. Cesàro spaces on $[0, 1]$ as interpolation spaces with respect to the couple $(L_1(1-t), Ces_\infty)$

In contrast to the case of the semi-axis  $[0, \infty)$ ,  $Ces_p[0, 1]$  for  $1 \leq p < \infty$  is not even an intermediate space between  $L_1[0, 1]$  and  $Ces_\infty[0, 1]$ . In fact,  $Ces_\infty[0, 1] \hookrightarrow L_1[0, 1]$ , but it is easy to show that  $Ces_p[0, 1] \not\subset L_1[0, 1]$  for every  $1 \leq p < \infty$ .

On the other hand, from the inequality  $1 - u \leq \ln 1/u$  ( $0 < u \leq 1$ ) it follows that  $Ces_p[0, 1]$ ,  $1 \leq p < \infty$ , is an intermediate space between the spaces  $L_1(1-t)[0, 1]$  and  $Ces_\infty[0, 1]$ , because of

$$Ces_\infty[0, 1] \xhookrightarrow{1} Ces_p[0, 1] \xhookrightarrow{1} Ces_1[0, 1] = L_1(\ln 1/t)[0, 1] \xhookrightarrow{1} L_1(1-t)[0, 1].$$

**THEOREM 2.** *If  $1 < p < \infty$ , then*

$$(Ces_1[0, 1], Ces_\infty[0, 1])_{1-1/p,p} \xhookrightarrow{1} Ces_p[0, 1] \quad (12)$$

and

$$(L_1(1-t)[0, 1], Ces_\infty[0, 1])_{1-1/p,p} = Ces_p[0, 1]. \quad (13)$$

*Proof.* All function spaces in this proof are considered on the segment  $I = [0, 1]$  if it is not indicated something different.

At first, for any  $f \in Ces_1$  and all  $0 < t \leq 1$  we have

$$K(t, f) := K(t, f; Ces_1, Ces_\infty) \geq \int_0^t (Cf)^*(s) ds. \quad (14)$$

In fact, we can assume that  $f \geq 0$ . If  $f = g + h$ ,  $g \geq 0$ ,  $h \geq 0$ ,  $g \in Ces_1$ ,  $h \in Ces_\infty$ , then  $Cf = Cg + Ch$  and, therefore, by formula (3),

$$\begin{aligned} \|g\|_{Ces(1)} + t \|h\|_{Ces(\infty)} &= \|Cg\|_{L_1} + t \|Ch\|_{L_\infty} \\ &\geq \inf\{\|y\|_{L_1} + t \|z\|_{L_\infty} : Cf = y + z, y \in L_1, z \in L_\infty\} \\ &= K(t, Cf; L_1, L_\infty) = \int_0^t (Cf)^*(s) ds. \end{aligned}$$

Taking the infimum over all suitable  $g$  and  $h$  we get (14). Next, by the definition of the real interpolation spaces, we obtain

$$\begin{aligned} \|f\|_{1-1/p,p}^p &\geq \int_0^1 [t^{1/p-1} K(t, f)]^p \frac{dt}{t} = \int_0^1 t^{-p} K(t, f)^p dt \\ &\geq \int_0^1 t^{-p} \left[ \int_0^t (Cf)^*(s) ds \right]^p dt \geq \|Cf\|_{L_p[0,1]}^p = \|f\|_{Ces(p)}^p, \end{aligned}$$

and the proof of imbedding (12) is complete.

Before to proceed with the proof of (13) we introduce the following notation: for a Banach function space  $E$  on  $I = [0, \infty)$  or  $[0, 1]$  and any set  $A \subset I$  by  $E|_A$  we will mean the subspace of  $E$ , which consists of all functions  $f$  such that  $\text{supp } f \subset A$ . Let us denote also  $X_p := (L_1(1-t), Ces_\infty)_{1-1/p,p}$ . Since

$$\|f\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{X_p} + \|f\chi_{[1/2,1]}\|_{X_p}.$$

then for proving (13) it is sufficient to check that

$$\|f\chi_{[0,1/2]}\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{Ces_p} \quad (15)$$

and

$$\|f\chi_{[1/2,1]}\|_{X_p} \approx \|f\chi_{[1/2,1]}\|_{Ces_p}. \quad (16)$$

Firstly, since  $L_1(1-t)|_{[0,1/2]} = L_1[0, \infty)|_{[0,1/2]}$  and  $Ces_\infty|_{[0,1/2]} = Ces_\infty[0, \infty)|_{[0,1/2]}$ , then, by Proposition 1 (see formula (10)), we obtain

$$\|f\chi_{[0,1/2]}\|_{X_p} \approx \|f\chi_{[0,1/2]}\|_{(L_1[0,\infty), Ces_\infty[0,\infty))_{1-1/p,p}} \approx \|f\chi_{[0,1/2]}\|_{Ces_p[0,\infty)}. \quad (17)$$

Note that

$$Ces_p[0, \infty)|_{[0,1/2]} = Ces_p[0, 1]|_{[0,1/2]} \quad (18)$$

with equivalence of norms. In fact, by [3, Remark 5],  $Ces_p[0, \infty)|_{[0,1]} = Ces_p \cap L_1$ . If  $\text{supp } g \subset [0, 1/2]$ , then we have

$$\|g\|_{L_1} = \int_0^{1/2} |g(s)| ds \leq 2^{1/p} \left( \int_{1/2}^1 \left( \frac{1}{t} \int_0^t |g(s)| ds \right)^p dt \right)^{1/p} \leq 2^{1/p} \|g\|_{Ces(p)}.$$

Combining this together with the previous equality, we obtain (18). From (18) and (17) it follows (15).

Now, we prove (16). Since  $(L_1(1-s)|_{[1/2,1]}, Ces_\infty|_{[1/2,1]})$  is a complemented subcouple of the Banach couple  $(L_1(1-s), Ces_\infty)$ , then, by the well-known result of Baouendi and

Goulaouic [4, Theorem 1] which is valid for arbitrary interpolation method (see also [23, Theorem 1.17.1]), we have

$$\|f\chi_{[1/2,1]}\|_{X_p} \approx \|f\chi_{[1/2,1]}\|_{Y_p},$$

where  $Y_p := (L_1(1-s)|_{[1/2,1]}, Ces_\infty|_{[1/2,1]})_{1-1/p,p}$ . Therefore, (16) will be proved whenever we show that

$$Y_p = Ces_p|_{[1/2,1]}. \quad (19)$$

On the one hand, since  $1-u \leq \ln 1/u \leq 2(1-u)$  for all  $1/2 \leq u \leq 1$  and  $Ces_1 = L_1(\ln 1/s)$ , then  $Ces_1|_{[1/2,1]} = L_1(1-s)|_{[1/2,1]}$ , and, by already proved imbedding (12), we obtain

$$Y_p = (Ces_1|_{[1/2,1]}, Ces_\infty|_{[1/2,1]})_{1-1/p,p} \subset Ces_p|_{[1/2,1]}.$$

For proving the opposite imbedding we note, firstly, that for any function  $h$  with  $\text{supp } h \subset [1/2, 1]$  we have

$$\|h\|_{Ces(\infty)} = \sup_{1/2 \leq x \leq 1} \frac{1}{x} \int_{1/2}^x |h(s)| ds,$$

whence

$$\|h\|_{L_1} = \int_{1/2}^1 |h(s)| ds \leq \|h\|_{Ces(\infty)} \leq 2 \int_{1/2}^1 |h(s)| ds = 2\|h\|_{L_1}.$$

Therefore, using formula for the  $K$ -functional with respect to a couple of weighted  $L_1$ -spaces (see (4)), we obtain

$$G(t, h) \leq K(t, h; L_1(1-s)|_{[1/2,1]}, Ces_\infty|_{[1/2,1]}) \leq 2G(t, h), \quad (20)$$

where

$$G(t, h) = K(t, h; L_1(1-s)|_{[1/2,1]}, L_1|_{[1/2,1]}) = \int_{1/2}^1 \min(1-s, t) |h(s)| ds.$$

Furthermore, let  $h \in L_1|_{[1/2,1]}$ . Then

$$Ch(s) = \frac{1}{s} \int_{1/2}^s |h(u)| du \geq \int_{1/2}^s |h(u)| du,$$

whence

$$(Ch)^*(s) \geq \int_{1/2}^{1-s} |h(u)| du, \quad 0 < s \leq 1.$$

Therefore, for all  $0 \leq t \leq 1$ , we obtain

$$\begin{aligned} \int_0^t (Ch)^*(s) ds &\geq \int_0^t \left( \int_{1/2}^{1-s} |h(u)| du \right) ds \\ &= \int_{1/2}^{1-t} \left( \int_0^t |h(u)| ds \right) du + \int_{1-t}^1 \left( \int_0^{1-u} |h(u)| ds \right) du \\ &= t \int_{1/2}^{1-t} |h(u)| du + \int_{1-t}^1 (1-u) |h(u)| du = G(t, h). \end{aligned}$$

From this inequality and the definition of  $G(t, h)$  it follows that the estimate

$$\int_0^{\min(1,t)} (Ch)^*(s) ds \geq G(t, h)$$

holds for all  $t > 0$ . Hence, by (20) and Hardy classical inequality for every  $h \in Ces_p$  with  $\text{supp } h \subset [1/2, 1]$ , we have

$$\begin{aligned} \|h\|_{Y_p} &= \left( \int_0^\infty t^{-p} K(t, h; L_1(1-s)|_{[1/2,1]}, Ces_\infty|_{[1/2,1]})^p dt \right)^{1/p} \\ &\leq 2 \left( \int_0^\infty t^{-p} G(t, h)^p dt \right)^{1/p} \leq 2 \left( \int_0^\infty t^{-p} \left( \int_0^{\min(1,t)} (Ch)^*(s) ds \right)^p dt \right)^{1/p} \\ &\leq 2 \left[ \left( \int_0^1 t^{-p} \left( \int_0^t (Ch)^*(s) ds \right)^p dt \right)^{1/p} + \left( \int_1^\infty t^{-p} \left( \int_0^1 (Ch)^*(s) ds \right)^p dt \right)^{1/p} \right] \\ &\leq 2 \left[ \frac{p}{p-1} \|Ch\|_{L_p[0,1]} + \frac{1}{(p-1)^{1/p}} \|Ch\|_{L_1[0,1]} \right] \leq \frac{4p}{p-1} \|h\|_{Ces_p[0,1]}. \end{aligned}$$

Thus,  $Ces_p|_{[1/2,1]} \subset Y_p$ , equality (19) holds, and the proof is complete.  $\square$

The following result is an immediate consequence of equality (13) and the reiteration equality (2).

**Corollary 3.** *If  $1 < p_0 < p_1 \leq \infty$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  with  $0 < \theta < 1$ , then*

$$(Ces_{p_0}[0, 1], Ces_{p_1}[0, 1])_{\theta, p} = Ces_p[0, 1].$$

**Remark 5.** An inspection of the proof of Theorem 2 shows that

$$(Ces_1|_{[1/2,1]}, Ces_\infty|_{[1/2,1]})_{1-1/p, p} = Ces_p|_{[1/2,1]}.$$

for every  $1 < p < \infty$  with equivalence of norms.

**Remark 6.** Comparison of formulas from Remark 4 and Corollary 3 shows that the real method  $(\cdot, \cdot)_{\theta, p}$  “well” interpolates the intersection of Cesàro spaces on the segment  $[0, 1]$  with the space  $L_1[0, 1]$  or, more precisely, we have

$$(Ces_{p_0}[0, 1] \cap L_1[0, 1], Ces_{p_1}[0, 1] \cap L_1[0, 1])_{\theta, p} = (Ces_{p_0}[0, 1], Ces_{p_1}[0, 1])_{\theta, p} \cap L_1[0, 1],$$

for all  $1 < p_0 < p_1 \leq \infty$ ,  $0 < \theta < 1$  and  $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ .

**Remark 7.** We will see further that imbedding (12) is strict for every  $1 < p < \infty$  and, even more, that  $Ces_p[0, 1]$  is not an interpolation space between the spaces  $Ces_1[0, 1]$  and  $Ces_\infty[0, 1]$ . Thus, the weighted  $L_1$ -space  $L_1(1-t)[0, 1]$  is in a sense the “proper” end of the scale of Cesàro spaces  $Ces_p[0, 1]$ ,  $1 < p \leq \infty$ .

## 5. The $K$ -functional for the couple $(Ces_1[0, 1], Ces_\infty[0, 1])$

In this part we will find an equivalent expression for the  $K$ -functional

$$K(t, f) = K(t, f; Ces_1, Ces_\infty) = K(t, f; Ces_1[0, 1], Ces_\infty[0, 1]).$$

We start with a lemma giving its lower estimate. Let us introduce two functions defined on  $(0, 1]$  by formulas

$$\tau_1(t) = t/\ln(e/t) \quad \text{and} \quad \tau_2(t) = e^{-t} \quad \text{for } 0 < t \leq 1. \quad (21)$$

It is easy to see that there exists a unique  $t_0 \in (0, 1)$  such that  $\tau_1(t_0) = \tau_2(t_0)$  and  $\tau_1(t) < \tau_2(t)$  if and only if  $0 < t < t_0$ .

**Lemma 1.** (lower estimates). *Let  $f \in Ces_1[0, 1]$ ,  $f \geq 0$  and  $0 < t \leq 1$ .*

(i) *If  $f_0 = f\chi_{[0, \tau_1(t)] \cup [\tau_2(t), 1]}$ , then*

$$K(t, f) \geq \frac{1}{4} \|f_0\|_{Ces(1)}. \quad (22)$$

(ii) *If  $f_1 = f\chi_{[\tau_1(t), \tau_2(t)]}$ , then*

$$K(t, f) \geq \frac{1}{e^2} t \|f_1\|_{Ces(\infty)}. \quad (23)$$

*Proof.* (i) Firstly, let us prove that

$$K(t, f) \geq \frac{1}{3} \|f\chi_{[0, \tau_1(t)]}\|_{Ces(1)} \quad \text{for all } 0 < t \leq 1. \quad (24)$$

Let  $f \in Ces_1$ ,  $f = g + h$ , where  $g \in Ces_1, h \in Ces_\infty$ . We may assume that  $f \geq 0$  and  $0 \leq g \leq f, 0 \leq h \leq f$ . Then

$$\begin{aligned} 3(\|g\|_{Ces(1)} + t\|h\|_{Ces(\infty)}) &\geq \|g\|_{Ces(1)} + 3t\|h\|_{Ces(\infty)} \\ &\geq \|(f - h)\chi_{[0, \tau_1(t)]}\|_{Ces(1)} + 3t\|h\chi_{[0, \tau_1(t)]}\|_{Ces(\infty)} \\ &= \|f\chi_{[0, \tau_1(t)]}\|_{Ces(1)} - \|h\chi_{[0, \tau_1(t)]}\|_{Ces(1)} \\ &\quad + 3t\|h\chi_{[0, \tau_1(t)]}\|_{Ces(\infty)}. \end{aligned} \quad (25)$$

Let us show that for any function  $v \in Ces_\infty$ ,  $v \geq 0$ , with  $\text{supp } v \subset [0, \tau_1(t)]$  we have

$$\|v\|_{Ces(1)} \leq 3t\|v\|_{Ces(\infty)}. \quad (26)$$

In fact, by the assumption on the support of  $v$  and by the Fubini theorem, we obtain

$$\begin{aligned} \|v\|_{Ces(1)} &= \int_0^{\tau_1(t)} \left( \frac{1}{s} \int_0^s v(u) du \right) ds + \int_{\tau_1(t)}^1 \left( \frac{1}{s} \int_0^{\tau_1(t)} v(u) du \right) ds \\ &= \int_0^{\tau_1(t)} \left( \frac{1}{s} \int_0^s v(u) du \right) ds + \int_0^{\tau_1(t)} \left( \int_{\tau_1(t)}^1 \frac{1}{s} ds \right) v(u) du \\ &= \int_0^{\tau_1(t)} \left( \frac{1}{s} \int_0^s v(u) du \right) ds + \int_0^{\tau_1(t)} v(u) du \ln \frac{1}{\tau_1(t)}. \end{aligned}$$

Since  $\tau_1(t) \leq t$  it follows that

$$\int_0^{\tau_1(t)} \left( \frac{1}{s} \int_0^s v(u) du \right) ds \leq \tau_1(t) \sup_{0 < s \leq \tau_1(t)} \frac{1}{s} \int_0^s v(u) du \leq t \|v\|_{Ces(\infty)}.$$

Moreover,

$$\begin{aligned} \int_0^{\tau_1(t)} v(u) du \ln \frac{1}{\tau_1(t)} &\leq \tau_1(t) \ln \frac{1}{\tau_1(t)} \sup_{0 < s \leq \tau_1(t)} \frac{1}{s} \int_0^s v(u) du \\ &= \frac{\ln \frac{1}{t} + \ln \ln \frac{e}{t}}{\ln \frac{e}{t}} t \|v\|_{Ces(\infty)} \leq 2t \|v\|_{Ces(\infty)}, \end{aligned}$$

and estimate (26) follows. Combining this estimate for  $v = h\chi_{[0, \tau_1(t)]}$  together with (25) we conclude that

$$3(\|g\|_{Ces(1)} + t\|h\|_{Ces(\infty)}) \geq \|f\chi_{[0, \tau_1(t)]}\|_{Ces(1)}.$$

Taking the infimum over all decompositions  $f = g + h, g \in Ces_1, h \in Ces_\infty$  with  $0 \leq g \leq f, 0 \leq h \leq f$  we obtain estimate (24).

Next, since  $Ces_1 = L_1(\ln \frac{1}{s})$  and  $Ces_\infty \xhookrightarrow{1} L_1$ , we have

$$K(t, f; L_1(\ln \frac{1}{s}), L_1) = K(t, f; Ces_1, L_1) \leq K(t, f).$$

Therefore, applying the well-known equality

$$K(t, f; L_1(\ln \frac{1}{s}), L_1) = \int_0^1 \min(\ln \frac{1}{s}, t) |f(s)| ds$$

and the elementary inequality

$$\int_0^1 \min(\ln \frac{1}{s}, t) |f(s)| ds \geq \int_{e^{-t}}^1 \ln \frac{1}{s} |f(s)| ds = \|f\chi_{[\tau_2(t), 1]}\|_{Ces(1)},$$

we obtain

$$K(t, f) \geq \|f\chi_{[\tau_2(t), 1]}\|_{Ces(1)}.$$

Inequality (22) is an immediate consequence of the last inequality and estimate (24). The proof of (i) is complete.

(ii) Since inequality (23) is obvious for  $t \in [t_0, 1]$ , it can be assumed that  $0 < t < t_0$ . Let again  $f \in Ces_1, f = g + h$ , where  $g \in Ces_1, h \in Ces_\infty$  and  $0 \leq g \leq f, 0 \leq h \leq f$ . Then for any  $c \in (0, 1)$  we have

$$\begin{aligned} \|g\|_{Ces(1)} + t\|h\|_{Ces(\infty)} &\geq \|g\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(1)} + ct\|(f - g)\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(\infty)} \\ &\geq \|g\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(1)} - ct\|g\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(\infty)} \\ &\quad + ct\|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(\infty)}. \end{aligned} \tag{27}$$

We want to show that for every positive function  $w \in Ces_1$  with  $\text{supp } w \subset [\tau_1(t), \tau_2(t)]$  the following inequality holds:

$$\frac{1}{e^2} t \|w\|_{Ces(\infty)} \leq \|w\|_{Ces(1)} \quad \text{for any } 0 < t < t_0. \tag{28}$$



Since

$$\begin{aligned}
\|w\|_{Ces(1)} &= \int_0^1 \frac{1}{s} \left( \int_{\tau_1(t)}^s w(u) du \cdot \chi_{[\tau_1(t), \tau_2(t)]}(s) + \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \chi_{[\tau_2(t), 1]}(s) \right) ds \\
&= \int_{\tau_1(t)}^{\tau_2(t)} \left( \frac{1}{s} \int_{\tau_1(t)}^s w(u) du \right) ds + \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \int_{\tau_2(t)}^1 \frac{ds}{s} \\
&= \int_{\tau_1(t)}^{\tau_2(t)} \left( \int_u^{\tau_2(t)} \frac{ds}{s} \right) w(u) du + \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \ln \frac{1}{\tau_2(t)} \\
&= \int_{\tau_1(t)}^{\tau_2(t)} w(u) \ln \frac{\tau_2(t)}{u} du + t \int_{\tau_1(t)}^{\tau_2(t)} w(u) du,
\end{aligned}$$

for proving the previous inequality it suffices to show that for all  $t \in (0, t_0)$  and  $s \in [\tau_1(t), \tau_2(t)]$  we have

$$\frac{1}{e^2} t \int_{\tau_1(t)}^s w(u) du \leq s \left( \int_{\tau_1(t)}^{\tau_2(t)} w(u) \ln \frac{\tau_2(t)}{u} du + t \int_{\tau_1(t)}^{\tau_2(t)} w(u) du \right). \quad (29)$$

We consider the cases when  $s \in [\tau_1(t), \frac{\tau_2(t)}{e}]$  and  $s \in (\frac{\tau_2(t)}{e}, \tau_2(t)]$  separately. Define a unique  $t_1 \in (0, t_0)$  such that  $\tau_1(t_1) = \frac{\tau_2(t_1)}{e}$  and note that the segment  $[\tau_1(t), \frac{\tau_2(t)}{e}]$  is non-empty only if  $0 < t \leq t_1$ . Let

$$\varphi(s) := s \cdot \ln \frac{\tau_2(t)}{s} \quad \text{for } s \in [\tau_1(t), \frac{\tau_2(t)}{e}].$$

Since  $\varphi'(s) = \ln \frac{\tau_2(t)}{s} - 1 = \ln \frac{\tau_2(t)}{es} \geq 0$  for all  $s \in [\tau_1(t), \frac{\tau_2(t)}{e}]$  it follows that  $\varphi$  increases. Therefore,  $\varphi(s) \geq \varphi(\tau_1(t))$  for all  $s \in [\tau_1(t), \frac{\tau_2(t)}{e}]$  and so

$$s \int_{\tau_1(t)}^{\tau_2(t)} w(u) \ln \frac{\tau_2(t)}{u} du \geq s \ln \frac{\tau_2(t)}{s} \int_{\tau_1(t)}^s w(u) du \geq \tau_1(t) \ln \frac{\tau_2(t)}{\tau_1(t)} \int_{\tau_1(t)}^s w(u) du. \quad (30)$$

We show that

$$\tau_1(t) \ln \frac{\tau_2(t)}{\tau_1(t)} \geq \frac{1}{e^2} t \quad \text{for all } 0 < t \leq t_1. \quad (31)$$

The function

$$\psi(t) = \frac{\tau_1(t)}{t} \ln \frac{\tau_2(t)}{\tau_1(t)} = \frac{\ln \frac{e}{t} - t}{\ln \frac{e}{t}} \quad \text{for } t \in (0, t_1]$$

has the derivative  $\psi'(t) = -[(t+1)(1 + \ln \frac{e}{t}) + \ln \tau_1(t)]/[t(\ln \frac{e}{t})^2]$ . It is not hard to check that  $\psi$  is increasing on  $(0, t_2)$  and decreasing on  $(t_2, t_1]$ , where a unique  $t_2 \in (0, t_1)$ . Hence, by the definition of  $t_1$ , for all  $t \in (0, t_1]$ , we have

$$\psi(t) \geq \min[\psi(0^+), \psi(t_1)] = \min \left( 1, \ln^{-1} \frac{e}{t_1} \right) = \ln^{-1} \frac{e}{t_1} = t_1^{-1} e^{-1-t_1} \geq e^{-2}.$$

Thus, we obtain inequality (31). Combining it with estimate (30), we obtain (29) in the case when  $0 < t \leq t_1$  and  $s \in [\tau_1(t), \frac{\tau_2(t)}{e}]$ .

In the second case, when  $s \in (\frac{\tau_2(t)}{e}, \tau_2(t)]$ , we have  $s \geq e^{-1-t} \geq e^{-2}$  and so

$$t \int_{\tau_1(t)}^s w(u) du \leq e^2 t s \int_{\tau_1(t)}^{\tau_2(t)} w(u) du.$$

Hence, estimate (29) holds again, and so inequality (28) is proved. Combining (28) and (27) with  $c = e^{-2}$ , we obtain

$$\|g\|_{Ces(1)} + t\|h\|_{Ces(\infty)} \geq \frac{1}{e^2} t \|f_1\|_{Ces(\infty)} \quad \text{for all } 0 < t < t_0.$$

Taking the infimum over all decompositions  $f = g + h, g \in Ces_1, h \in Ces_\infty$  with  $0 \leq g \leq f, 0 \leq h \leq f$  we come to estimate (23), and the proof of (ii) is complete.  $\square$

**THEOREM 3.** *For every function  $f \in Ces_1[0, 1]$  we have*

$$\begin{aligned} \frac{1}{2e^2} (\|f\chi_{[0, \tau_1(t)] \cup [\tau_2(t), 1]}\|_{Ces(1)} + t \|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(\infty)}) \\ \leq K(t, f; Ces_1, Ces_\infty) \\ \leq \|f\chi_{[0, \tau_1(t)] \cup [\tau_2(t), 1]}\|_{Ces(1)} + t \|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(\infty)}, \end{aligned}$$

for all  $0 < t < 1$ , and  $K(t, f; Ces_1, Ces_\infty) = \|f\|_{Ces(1)}$  for all  $t \geq 1$ .

*Proof.* The first inequality is a consequence of Lemma 1 and the definition of the  $K$ -functional. The equality  $K(t, f; Ces_1, Ces_\infty) = \|f\|_{Ces(1)}$  ( $t \geq 1$ ) follows from the imbedding  $Ces_\infty \xhookrightarrow{1} Ces_1$ .  $\square$

If a positive function  $f \in Ces_1[0, 1]$  is decreasing, then the description of the  $K$ -functional can be simplified.

**THEOREM 4.** *If  $f \in Ces_1[0, 1], f \geq 0$  and  $f$  is decreasing, then*

$$\frac{1}{3} \|f\chi_{[0, \tau_1(t)]}\|_{Ces(1)} \leq K(t, f; Ces_1, Ces_\infty) \leq \|f\chi_{[0, \tau_1(t)]}\|_{Ces(1)} \quad (32)$$

for all  $0 < t < 1$  and  $K(t, f; Ces_1, Ces_\infty) = \|f\|_{Ces(1)}$  for all  $t \geq 1$ .

*Proof.* Taking into account the proof of Lemma 1(i) (see inequality (24)) it suffices to prove only the right-hand side inequality in (32).

Let  $f_0 := [f - f(\tau_1(t))]\chi_{[0, \tau_1(t)]}$  and  $f_1 := f - f_0$ . Since  $f \geq 0$  is decreasing, we have  $\|f_1\|_{Ces(\infty)} = f(\tau_1(t))$ . Therefore, by Fubini theorem,

$$\begin{aligned} \|f_0\|_{Ces(1)} + t \|f_1\|_{Ces(\infty)} &= \int_0^1 \frac{1}{s} \int_0^s (f(u) - f(\tau_1(t))) \chi_{[0, \tau_1(t)]}(u) du ds + t f(\tau_1(t)) \\ &= \int_0^1 \frac{1}{s} \int_0^s f(u) \chi_{[0, \tau_1(t)]}(u) du ds - f(\tau_1(t)) \int_0^{\tau_1(t)} \ln \frac{1}{u} du + t f(\tau_1(t)) \end{aligned}$$

$$\begin{aligned}
&= \|f\chi_{[0,\tau_1(t)]}\|_{Ces(1)} - f(\tau_1(t))\tau_1(t) \left[1 + \ln \frac{1}{\tau_1(t)}\right] + t f(\tau_1(t)) \\
&= \|f\chi_{[0,\tau_1(t)]}\|_{Ces(1)} + t f(\tau_1(t)) \left[1 - \frac{1 + \ln(\ln \frac{e}{t}/t)}{\ln \frac{e}{t}}\right] \\
&= \|f\chi_{[0,\tau_1(t)]}\|_{Ces(1)} - \frac{t f(\tau_1(t)) \ln(\ln \frac{e}{t})}{\ln \frac{e}{t}} \leq \|f\chi_{[0,\tau_1(t)]}\|_{Ces(1)},
\end{aligned}$$

whence

$$K(t, f; Ces_1, Ces_\infty) \leq \|f\chi_{[0,\tau_1(t)]}\|_{Ces(1)},$$

and the desired result is proved.  $\square$

## 6. Identification of the real interpolation spaces $(Ces_1[0, 1], Ces_\infty[0, 1])_{1-1/p, p}$ for $1 < p < \infty$

Let us define the weighted Cesàro function space  $Ces_p(\ln \frac{e}{t})[0, 1]$  consisting of all Lebesgue measurable functions  $f$  on  $[0, 1]$  such that

$$\|f\|_{Ces(p, \ln)} := \left( \int_0^1 \left( \frac{1}{x} \int_0^x |f(t)| dt \right)^p \ln \frac{e}{x} dx \right)^{1/p} < \infty.$$

Clearly,  $Ces_p(\ln \frac{e}{t})[0, 1] \xrightarrow{1} Ces_p[0, 1]$  for every  $1 < p < \infty$ , and this imbedding is strict.

**THEOREM 5.** For  $1 < p < \infty$

$$(Ces_1[0, 1], Ces_\infty[0, 1])_{1-1/p, p} = Ces_p(\ln \frac{e}{t})[0, 1]. \quad (33)$$

*Proof.* Denote  $X_p = (Ces_1, Ces_\infty)_{1-1/p, p}$ ,  $1 < p < \infty$ . Using Theorem 3 on the  $K$ -functional for the couple  $(Ces_1, Ces_\infty)$  on  $[0, 1]$ , we have

$$\begin{aligned}
\|f\|_{X_p} &\leq \left[ \int_0^{t_0} t^{-p} \|f\chi_{[0,\tau_1(t)]}\|_{Ces(1)}^p dt \right]^{1/p} + \left[ \int_0^{t_0} t^{-p} \|f\chi_{[\tau_2(t), 1]}\|_{Ces(1)}^p dt \right]^{1/p} \\
&+ \left[ \int_0^{t_0} t^{-p} (t \|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(\infty)})^p dt \right]^{1/p} + \left[ \int_{t_0}^\infty t^{-p} \|f\|_{Ces(1)}^p dt \right]^{1/p} \\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \left[ \int_0^{t_0} t^{-p} \left( \int_0^{\tau_1(t)} C f(s) ds + \int_{\tau_1(t)}^1 C(f\chi_{[0,\tau_1(t)]})(s) ds \right)^p dt \right]^{1/p} \\
&\leq \left[ \int_0^{t_0} t^{-p} \left( \int_0^{\tau_1(t)} C f(s) ds \right)^p dt \right]^{1/p} + \left[ \int_0^{t_0} t^{-p} \left( \int_{\tau_1(t)}^1 C(f\chi_{[0,\tau_1(t)]})(s) ds \right)^p dt \right]^{1/p} \\
&= I_{11} + I_{12}.
\end{aligned}$$

First of all, we estimate all five integrals from above. Since  $\tau_1'(t) = (\ln \frac{e}{t} + 1)/(\ln \frac{e}{t})^2$  and so  $1/\ln(e/t) \leq \tau_1'(t) \leq 2/\ln(e/t)$  for all  $0 < t \leq 1$ , we get

$$\begin{aligned} I_{11}^p &\leq \int_0^{t_0} t^{-p} (\ln \frac{e}{t})^{p-1} \left( \int_0^{\tau_1(t)} C f(s) ds \right)^p dt \\ &\leq \int_0^{t_0} \tau_1(t)^{-p} \left( \int_0^{\tau_1(t)} C f(s) ds \right)^p d\tau_1(t). \end{aligned}$$

Putting  $u = \tau_1(t)$  and using classical Hardy inequality, we obtain

$$\begin{aligned} I_{11} &\leq \left[ \int_0^{\tau_1(t_0)} \left( \frac{1}{u} \int_0^u C f(s) ds \right)^p du \right]^{1/p} \leq \|C^2 f\|_{L_p[0,1]} \\ &\leq p' \|C f\|_{L_p[0,1]} = p' \|f\|_{Ces(p)} \leq p' \|f\|_{Ces(p, \ln)}. \end{aligned}$$

Next, by the estimate  $\ln \frac{1}{\tau_1(t)} \leq 2 \ln \frac{e}{t}$ ,  $0 < t \leq 1$ , we get

$$\begin{aligned} I_{12}^p &= \int_0^{t_0} t^{-p} \left( \int_{\tau_1(t)}^1 \left( \frac{1}{s} \int_0^{\tau_1(t)} |f(u)| du \right) ds \right)^p dt \\ &= \int_0^{t_0} t^{-p} \left( \int_0^{\tau_1(t)} |f(u)| du \right)^p \ln^p \frac{1}{\tau_1(t)} dt \\ &\leq 2^p \int_0^{t_0} \tau_1(t)^{-p} \left( \int_0^{\tau_1(t)} |f(u)| du \right)^p dt. \end{aligned}$$

Substitution  $t = \tau_1^{-1}(s)$  and the inequalities

$$(\tau_1^{-1})'(s) = \frac{1}{\tau_1'(\tau_1^{-1}(s))} \leq \ln \frac{e}{\tau_1^{-1}(s)} \leq \ln \frac{e}{s} \quad (34)$$

show that

$$\begin{aligned} I_{12} &\leq 2 \left[ \int_0^{\tau_1(t_0)} \left( \frac{1}{s} \int_0^s |f(u)| du \right)^p \ln \frac{e}{s} ds \right]^{1/p} \\ &\leq 2 \left[ \int_0^1 (C f(s))^p \ln \frac{e}{s} ds \right]^{1/p} = 2 \|f\|_{Ces(p, \ln)}. \end{aligned}$$

From the equality  $Ces_1[0, 1] = L_1(\ln 1/u)$  and the inequalities  $\ln 1/u \leq e(1-u)$  ( $1/e \leq u \leq 1$ ) and  $\tau_2(t) = e^{-t} \geq 1-t$  ( $0 < t \leq 1$ ) it follows

$$\begin{aligned} I_2^p &= \int_0^{t_0} t^{-p} \|f \chi_{[\tau_2(t), 1]}\|_{Ces(1)}^p dt = \int_0^{t_0} t^{-p} \left( \int_{\tau_2(t)}^1 |f(u)| \ln \frac{1}{u} du \right)^p dt \\ &\leq e^p \int_0^{t_0} t^{-p} \left( \int_{\tau_2(t)}^1 |f(u)|(1-u) du \right)^p dt \\ &\leq e^p \int_0^{t_0} t^{-p} \left( \int_{1-t}^1 |f(u)|(1-u) du \right)^p dt. \end{aligned}$$

Arguing in the same way as in the second part of the proof of Theorem 2, for  $g = f\chi_{[e^{-1},1]}$  and  $0 < s \leq 1$  we have

$$Cg(s) = \frac{1}{s} \int_{e^{-1}}^s |f(u)| du \geq \int_{e^{-1}}^s |f(u)| du,$$

whence  $(Cg)^*(s) \geq \int_{e^{-1}}^{1-s} |f(u)| du$  and

$$\begin{aligned} \int_0^t (Cg)^*(s) ds &\geq \int_0^t \left( \int_{e^{-1}}^{1-s} |f(u)| du \right) ds = \int_{e^{-1}}^{1-t} \left( \int_0^t |f(u)| ds \right) du \\ &+ \int_{1-t}^1 \left( \int_0^{1-u} |f(u)| ds \right) du \geq \int_{1-t}^1 |f(u)|(1-u) du. \end{aligned}$$

Therefore, again by the classical Hardy inequality,

$$\begin{aligned} I_2 &\leq e \left[ \int_0^{t_0} t^{-p} \left( \int_0^t (Cg)^*(s) ds \right)^p dt \right]^{1/p} \\ &\leq e \|C[(Cg)^*]\|_{L_p[0,1]} \leq ep' \|(Cg)^*\|_{L_p[0,1]} \\ &= ep' \|Cg\|_{L_p[0,1]} = ep' \|f\chi_{[e^{-1},1]}\|_{Ces(p)} \\ &\leq ep' \|f\|_{Ces(p)} \leq ep' \|f\|_{Ces(p, \ln)}. \end{aligned}$$

For the third integral, we have

$$\begin{aligned} I_3 &= \left[ \int_0^{t_0} \|f\chi_{[\tau_1(t), \tau_2(t)]}\|_{Ces(\infty)}^p dt \right]^{1/p} \\ &\leq \left[ \int_0^{t_0} \sup_{\tau_1(t) < s \leq 1/2} \left( \frac{1}{s} \int_0^s |f(u)\chi_{[\tau_1(t), \tau_2(t)]}(u)| du \right)^p dt \right]^{1/p} \\ &+ \left[ \int_0^{t_0} \sup_{1/2 < s \leq \tau_2(t)} \left( \frac{1}{s} \int_0^s |f(u)\chi_{[\tau_1(t), \tau_2(t)]}(u)| du \right)^p dt \right]^{1/p} \\ &= \left[ \int_0^{t_0} \sup_{\tau_1(t) < s \leq 1/2} \left( \frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du \right)^p dt \right]^{1/p} \\ &+ \left[ \int_0^{t_0} \sup_{1/2 < s \leq \tau_2(t)} \left( \frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du \right)^p dt \right]^{1/p} = I_{31} + I_{32}. \end{aligned}$$

If  $\tau_1(t) < s \leq 1/2$ , then  $2s \leq 1$  and

$$\begin{aligned} \int_{\tau_1(t)}^{2s} \left( \frac{1}{v} \int_0^v |f(u)| du \right) dv &= \int_0^{\tau_1(t)} \left( \int_{\tau_1(t)}^{2s} \frac{1}{v} dv \right) |f(u)| du + \int_{\tau_1(t)}^{2s} \left( \int_u^{2s} \frac{1}{v} dv \right) |f(u)| du \\ &= \int_0^{\tau_1(t)} |f(u)| du \ln \frac{2s}{\tau_1(t)} + \int_{\tau_1(t)}^{2s} |f(u)| \ln \frac{2s}{u} du \\ &\geq \frac{2s - \tau_1(t)}{2s} \int_{\tau_1(t)}^{2s} |f(u)| \ln \frac{2s}{u} du \\ &\geq \ln 2 \frac{2s - \tau_1(t)}{2s} \int_{\tau_1(t)}^s |f(u)| du. \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{\tau_1(t) < s \leq 1/2} \frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du &\leq \frac{2}{\ln 2} \sup_{\tau_1(t) < s \leq 1/2} \frac{1}{2s - \tau_1(t)} \int_{\tau_1(t)}^{2s} Cf(v) dv \\ &\leq \frac{2}{\ln 2} MCf(\tau_1(t)), \end{aligned}$$

where  $M$  is the maximal Hardy-Littlewood operator on  $[0, 1]$ . The above estimates show that

$$I_{31} \leq \frac{2}{\ln 2} \left( \int_0^{t_0} MCf(\tau_1(t))^p dt \right)^{1/p}.$$

Using once again substitution  $t = \tau_1^{-1}(s)$  and estimates (34), we obtain

$$I_{31} \leq \frac{2}{\ln 2} \left[ \int_0^{\tau_1(t_0)} [MCf(s)]^p \ln \frac{e}{s} ds \right]^{1/p} \leq \frac{2}{\ln 2} \|MCf\|_{L_p(\ln \frac{e}{s})}.$$

We will show in the next lemma that the maximal operator  $M$  is bounded in  $L_p(\ln \frac{e}{s})[0, 1]$  for  $1 < p < \infty$ , which implies that for some constant  $B_p \geq 1$ , which depends only on  $p$ , we have

$$I_{31} \leq \frac{2B_p}{\ln 2} \|Cf\|_{L_p(\ln \frac{e}{s})} = \frac{2B_p}{\ln 2} \|f\|_{Ces(p, \ln)}.$$

For the second part of the integral  $I_3$  we estimate in the following way:

$$\begin{aligned} I_{32}^p &= \int_0^{t_0} \sup_{1/2 < s \leq \tau_2(t)} \left( \frac{1}{s} \int_{\tau_1(t)}^s |f(u)| du \right)^p dt \leq 2^p \int_0^{t_0} \left( \int_{\tau_1(t)}^{\tau_2(t)} |f(u)| du \right)^p dt \\ &\leq 2^p \int_0^{t_0} \left( \frac{1}{\tau_2(t)} \int_0^{\tau_2(t)} |f(u)| du \right)^p dt, \end{aligned}$$

and, changing variable  $s = \tau_2(t) = e^{-t}$ , we obtain

$$\begin{aligned} I_{32} &\leq 2 \left[ \int_{e^{-t_0}}^1 \left( \frac{1}{s} \int_0^s |f(u)| du \right)^p \frac{ds}{s} \right]^{1/p} \leq 2e^{t_0/p} \left( \int_0^1 Cf(s)^p ds \right)^{1/p} \\ &\leq 2e \|f\|_{Ces(p)} \leq 2e \|f\|_{Ces(p, \ln)}. \end{aligned}$$

Since  $t_0 > 1/2$ , for the last integral we have

$$I_4 = \frac{1}{(p-1)^{1/p} t_0^{1-1/p}} \|f\|_{Ces(1)} \leq \frac{2}{p-1} \|f\|_{Ces(1)} \leq \frac{2}{p-1} \|f\|_{Ces(p, \ln)}.$$

Finally, summing up the above estimates, we get  $\|f\|_{X_p} \leq C_p \|f\|_{Ces(p, \ln)}$ , where  $C_p$  depends only on  $p$ . Thus, the imbedding  $Ces(p, \ln) \hookrightarrow X_p$  is proved.

Now, we proceed with estimations from below. Firstly, by inequality (24), we have

$$\|f\|_{X_p}^p \geq 3^{-p} \int_0^{t_0} t^{-p} \|f\chi_{[0, \tau_1(t)]}\|_{Ces(1)}^p dt = 3^{-p} I_1^p \geq 3^{-p} I_{12}^p. \quad (35)$$

It is not hard to check that  $\ln \frac{1}{\tau_1(t)} = \ln \frac{\ln(e/t)}{t} \geq e^{-1} \ln \frac{e}{t}$  for  $t \in (0, t_0]$ . Therefore,

$$I_{12}^p = \int_0^{t_0} t^{-p} \left( \int_0^{\tau_1(t)} |f(u)| du \right)^p \ln^p \frac{1}{\tau_1(t)} dt \geq e^{-p} \int_0^{t_0} \tau_1(t)^{-p} \left( \int_0^{\tau_1(t)} |f(u)| du \right)^p dt.$$

Since  $\tau_1'(s) \leq 2/\ln(e/s)$ ,  $\tau_1^{-1}(s) \leq s \ln(e/s)$  and  $\ln \ln(e/s) \leq e^{-1} \ln(e/s)$  ( $0 < s \leq 1$ ), we have

$$\begin{aligned} (\tau_1^{-1})'(s) &= \frac{1}{\tau_1'(\tau_1^{-1}(s))} \geq \frac{1}{2} \ln \frac{e}{\tau_1^{-1}(s)} \geq \frac{1}{2} \ln \frac{e}{s \ln(e/s)} \\ &= \frac{1}{2} (\ln \frac{e}{s} - \ln \ln \frac{e}{s}) \geq \frac{1}{2} (1 - \frac{1}{e}) \ln \frac{e}{s}. \end{aligned}$$

Hence, after substitution  $t = \tau_1^{-1}(s)$ , we obtain

$$I_{12}^p \geq e^{-p} \frac{1}{2} (1 - \frac{1}{e}) \int_0^{\tau_1(t_0)} \left( \frac{1}{s} \int_0^s |f(u)| du \right)^p \ln \frac{e}{s} ds \geq \frac{1}{4} e^{-p} \int_0^{\tau_1(t_0)} C f(s)^p \ln \frac{e}{s} ds,$$

and so, taking into account (35), we get

$$\|f\|_{X_p}^p \geq 4^{-1} (3e)^{-p} \int_0^{\tau_1(t_0)} C f(s)^p \ln \frac{e}{s} ds.$$

On the other hand, by the definition of  $t_0$ ,

$$\begin{aligned} \int_{\tau_1(t_0)}^1 C f(s)^p \ln \frac{e}{s} ds &\leq \ln \frac{e}{\tau_1(t_0)} \int_{\tau_1(t_0)}^1 C f(s)^p ds \leq (1 + t_0) \|f\|_{Ces(p)}^p \\ &\leq 2 \|f\|_{Ces(p)}^p \leq 2 \|f\|_{X_p}^p, \end{aligned}$$

where the last inequality follows from imbedding (12) of Theorem 2. Hence,

$$\|f\|_{X_p} \geq 8^{-1/p} (3e)^{-1} \left( \int_0^1 C f(s)^p \ln \frac{e}{s} ds \right)^{1/p} \geq \frac{1}{72} \|f\|_{Ces(p, \ln)},$$

and the imbedding  $X_p \hookrightarrow Ces(p, \ln)$  is proved. Thus, the proof of Theorem 5 will be finished if we show that the following lemma holds.  $\square$

**Lemma 2.** *If  $1 < p < \infty$ , then the maximal Hardy-Littlewood operator  $M$  on  $[0, 1]$  is bounded in the weighted space  $L_p(\ln \frac{e}{x})[0, 1] = L_p([0, 1], \ln \frac{e}{x} dx)$ .*

*Proof.* Muckenhoupt [17, Theorem 2] proved that the maximal operator  $M$  on  $[0, 1]$  is bounded in  $L_p([0, 1], w(x) dx)$  if and only if the weight  $w(x)$  satisfies the so-called  $A_p$ -condition on  $[0, 1]$ , that is,

$$\sup_{(a,b) \subset [0,1]} \left( \frac{1}{b-a} \int_a^b w(x) dx \right) \left( \frac{1}{b-a} \int_a^b w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty.$$

Therefore, it is enough to show that for all intervals  $(a, b) \subset [0, 1]$  we have

$$\int_a^b \ln \frac{e}{x} dx \left( \int_a^b \left( \ln \frac{e}{x} \right)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq 2(b-a)^p. \quad (36)$$

Note that for  $t \in (0, b)$

$$\int_t^b \ln \frac{e}{x} dx = b \ln \frac{e}{b} - t \ln \frac{e}{t} + b - t$$

and

$$\begin{aligned} \int_t^b \left( \ln \frac{e}{x} \right)^{-\alpha} dx &= b \left( \ln \frac{e}{b} \right)^{-\alpha} - t \left( \ln \frac{e}{t} \right)^{-\alpha} - \alpha \int_t^b \left( \ln \frac{e}{x} \right)^{-\alpha-1} dx \\ &\leq b \left( \ln \frac{e}{b} \right)^{-\alpha} - t \left( \ln \frac{e}{t} \right)^{-\alpha}, \end{aligned}$$

where  $\alpha > 0$ . Since the functions

$$\varphi_1(t) = \frac{b \ln(e/b) - t \ln(e/t) + b - t}{b - t} \quad \text{and} \quad \varphi_2(t) = \frac{b \left( \ln(e/b) \right)^{-\alpha} - t \left( \ln(e/t) \right)^{-\alpha}}{b - t}$$

are both decreasing on the interval  $(0, b)$  for every  $0 < b \leq 1$  it follows that  $\max_{0 < t < b} \varphi_1(t) = \varphi_1(0^+) = \ln(e^2/b)$  and  $\max_{0 < t < b} \varphi_2(t) = \varphi_2(0^+) = \ln^{-\alpha}(e/b)$ . Therefore, setting  $\alpha = \frac{1}{p-1}$ , for arbitrary  $0 \leq a < b \leq 1$  we have

$$\frac{1}{(b-a)^p} \int_a^b \ln \frac{e}{x} dx \left( \int_a^b \left( \ln \frac{e}{x} \right)^{-\frac{1}{p-1}} dx \right)^{p-1} \leq \ln \frac{e^2}{b} \left( \left( \ln \left( \frac{e}{b} \right) \right)^{-1/(p-1)} \right)^{p-1} = \frac{\ln(e^2/b)}{\ln(e/b)} \leq 2,$$

and inequality (36) is proved.  $\square$

## 7. $Ces_p[0, 1]$ , $1 < p < \infty$ , is not an interpolation space between $Ces_1[0, 1]$ and $Ces_\infty[0, 1]$

We start with two lemmas (it is instructive to compare the result from the first of them with imbedding (12)).

**Lemma 3.** *If  $1 < p < \infty$ , then*

$$Ces_p[0, 1] \not\hookrightarrow (Ces_1[0, 1], Ces_\infty[0, 1])_{1-1/p, \infty}. \quad (37)$$

*Proof.* Let us consider the family of characteristic functions  $f_s = \chi_{[0, s]}$ ,  $0 < s < 1$ . As we know (cf. Theorem 4),

$$K(t, f_s; Ces_1, Ces_\infty) \geq \frac{1}{3} \|f_s \chi_{[0, \tau_1(t)]}\|_{Ces(1)} \quad \text{for all } t > 0.$$

Since

$$\begin{aligned} \|f_s \chi_{[0, \tau_1(t)]}\|_{Ces(1)} &= \|\chi_{[0, \min(s, \tau_1(t))]} \|_{Ces(1)} = \|\chi_{[0, \min(s, \tau_1(t))]} \|_{L_1(\ln 1/s)} \\ &= \int_0^{\min(s, \tau_1(t))} \ln \frac{1}{s} ds = \min(s, \tau_1(t)) \left( \ln \frac{1}{\min(s, \tau_1(t))} + 1 \right), \end{aligned}$$



it follows that for all  $t$  such that  $\tau_1(t) \leq s$  we have  $\|f_s \chi_{[0, \tau_1(t)]}\|_{Ces(1)} \geq \tau_1(t) \ln \frac{1}{\tau_1(t)}$ . Therefore, using the inequality  $\tau_1^{-1}(s) \leq s \ln(e/s)$  once again, for  $0 < s < e^{-1}$  we obtain

$$\begin{aligned} \|f_s\|_{(Ces_1, Ces_\infty)_{1-1/p, \infty}} &= \sup_{t>0} t^{1/p-1} K(t, f_s; Ces_1, Ces_\infty) \\ &\geq \frac{1}{3} \sup_{t>0, \tau_1(t) \leq s} t^{1/p-1} \tau_1(t) \ln \frac{1}{\tau_1(t)} \\ &\geq \frac{1}{3} (\tau_1^{-1}(s))^{1/p-1} s \ln \frac{1}{s} \geq \frac{1}{6} (s \ln \frac{e}{s})^{1/p-1} s \ln \frac{1}{s} \\ &\geq \frac{1}{6} s^{1/p} (\ln \frac{e}{s})^{1/p}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \|f_s\|_{Ces_p} &= \left[ \int_0^s \left( \frac{1}{u} \int_0^u \chi_{[0,s]}(v) dv \right)^p du + \int_s^1 \left( \frac{1}{u} \int_0^u \chi_{[0,s]}(v) dv \right)^p du \right]^{1/p} \\ &= \left( s + s^p \int_s^1 u^{-p} du \right)^{1/p} = \left( s + \frac{s^p}{p-1} (s^{1-p} - 1) \right)^{1/p} \\ &= \left( \frac{p}{p-1} s - \frac{1}{p-1} s^p \right)^{1/p} \leq (p')^{1/p} s^{1/p}. \end{aligned}$$

Therefore, for  $0 < s < e^{-1}$

$$\frac{\|f_s\|_{(Ces_1, Ces_\infty)_{1-1/p, \infty}}}{\|f_s\|_{Ces_p}} \geq \frac{\frac{1}{6} s^{1/p} (\ln \frac{e}{s})^{1/p}}{(p')^{1/p} s^{1/p}} \geq \frac{1}{6p'} (\ln \frac{e}{s})^{1/p},$$

whence

$$\sup_{0 < s < 1} \frac{\|f_s\|_{(Ces_1, Ces_\infty)_{1-1/p, \infty}}}{\|f_s\|_{Ces_p}} = \infty,$$

which shows that (37) holds.  $\square$

Recall that the *characteristic function*  $\varphi(s, t)$  of an exact interpolation functor  $\mathcal{F}$  is defined by the equality  $\mathcal{F}(s\mathbb{R}, t\mathbb{R}) = \varphi(s, t) \mathbb{R}$  for all  $s, t > 0$ . By the Aronszajn–Gagliardo theorem (see [7, Theorem 2.5.1] or [8, Theorem 2.3.15]), for arbitrary Banach couple  $(X_0, X_1)$  and for every Banach space  $X \in \text{Int}(X_0, X_1)$  there is an exact interpolation functor  $\mathcal{F}$  such that  $\mathcal{F}(X_0, X_1) = X$ .

**Lemma 4.** *Let  $1 < p < \infty$ . Suppose that the Cesàro space  $Ces_p[0, 1] \in \text{Int}(Ces_1[0, 1], Ces_\infty[0, 1])$  and  $\mathcal{F}$  is an exact interpolation functor such that*

$$\mathcal{F}(Ces_1[0, 1], Ces_\infty[0, 1]) = Ces_p[0, 1]. \quad (38)$$

*Then the characteristic function  $\varphi(1, t)$  of  $\mathcal{F}$  is equivalent to  $t^{1/p}$  for  $0 < t \leq 1$ .*

*Proof.* To simplify notation let us denote  $V_p := Ces_p|_{[1/2, 1]}$  ( $1 \leq p \leq \infty$ ), that is,  $V_p$  is the subspace of  $Ces_p[0, 1]$ , which consists of all functions  $f$  such that  $\text{supp } f \subset [\frac{1}{2}, 1]$ . Since

$(V_1, V_\infty)$  is a complemented couple of the Banach couple  $(Ces_1[0, 1], Ces_\infty[0, 1])$ , by (38) and equality in Remark 5, we obtain

$$\mathcal{F}(V_1, V_\infty) = V_p = (V_1, V_\infty)_{1-1/p, p}. \quad (39)$$

Consider the sequence of functions  $g_k(t) = \chi_{[1-2^{-k}, 1-2^{-k-1}]}(t)$ ,  $k = 1, 2, \dots$  and the linear projection

$$Pf(t) = \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} f(s) ds \cdot g_k(t), \quad f \in V_\infty.$$

We have

$$\begin{aligned} \|Pf\|_{V_\infty} &\leq 2 \|Pf\|_{L_1[1/2, 1]} \leq 2 \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \cdot 2^{-k-1} \\ &= 2 \|f\|_{L_1[1/2, 1]} \leq 2 \|f\|_{V_\infty}, \end{aligned}$$

and, since  $1 - u \leq \ln(1/u) \leq 2(1 - u)$  for  $1/2 \leq u \leq 1$ ,

$$\begin{aligned} \|Pf\|_{V_1} &\leq \sum_{k=1}^{\infty} 2^{k+1} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} \ln \frac{1}{t} dt \\ &\leq \sum_{k=1}^{\infty} 2^{k+2} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} (1-t) dt \\ &\leq \sum_{k=1}^{\infty} 2^{k+2} \cdot 2^{-2k-1} \cdot \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| ds \\ &\leq 4 \sum_{k=1}^{\infty} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)|(1-s) ds \\ &\leq 4 \sum_{k=1}^{\infty} \int_{1-2^{-k}}^{1-2^{-k-1}} |f(s)| \ln \frac{1}{s} ds = 4 \|f\|_{L_1(\ln 1/s)} = 4 \|f\|_{V_1}. \end{aligned}$$

Therefore,  $P$  is a bounded linear projection from  $V_\infty$  onto  $\text{Im } P|_{V_\infty}$  and from  $V_1$  onto  $\text{Im } P|_{V_1}$ . At the same time, it is easy to see that the sequence  $\{2^{k+1} g_k\}_{k=1}^{\infty}$  is equivalent in  $V_\infty$  (resp. in  $V_1$ ) to the standard basis in  $l_1$  (resp. in  $l_1(2^{-k})$ ). Hence,  $(l_1, l_1(2^{-k}))$  is a complemented subcouple of the Banach couple  $(V_1, V_\infty)$  and therefore, by (35) and by the Baouendi-Goulaouic result [4, Theorem 1] (see also [23, Theorem 1.17.1]),

$$\mathcal{F}(l_1, l_1(2^{-k})) = (l_1, l_1(2^{-k}))_{1-1/p, p}.$$

In particular, from the last relation it follows that

$$\mathcal{F}(\mathbb{R}, 2^{-k} \mathbb{R}) = (\mathbb{R}, 2^{-k} \mathbb{R})_{1-1/p, p} = 2^{-k/p} \mathbb{R}$$

uniformly in  $k \in \mathbb{N}$ . Since the characteristic function of any exact interpolation functor is quasi-concave [8, Proposition 2.3.10], this implies the result.  $\square$

**THEOREM 6.** *For any  $1 < p < \infty$  the space  $Ces_p[0, 1]$  is not an interpolation space between the spaces  $Ces_1[0, 1]$  and  $Ces_\infty[0, 1]$ .*

*Proof.* Assume that  $Ces_p[0, 1]$  is an interpolation space between  $Ces_1[0, 1]$  and  $Ces_\infty[0, 1]$ . Then there is an exact interpolation functor  $\mathcal{F}$  such that equality (38) holds. By Lemma 4, the characteristic function  $\varphi(1, t)$  of  $\mathcal{F}$  is equivalent to  $t^{1/p}$  for  $0 < t \leq 1$ . Therefore, for any Banach couple  $(X_0, X_1)$  we have

$$\mathcal{F}(X_0, X_1) \subset (X_0, X_1)_{\psi, \infty}, \quad (40)$$

where  $(X_0, X_1)_{\psi, \infty}$  is the real interpolation space consisting of all  $x \in X_0 + X_1$  such that  $\sup_{t>0} \frac{\psi(t)}{t} K(t, x; X_0, X_1) < \infty$  and  $\psi(t) = \min(1, t^{1/p})$  [8, Proposition 3.8.6]. Since  $Ces_\infty[0, 1] \xrightarrow{1} Ces_1[0, 1]$ , then applying (40) to the couple  $(Ces_1[0, 1], Ces_\infty[0, 1])$ , we obtain

$$\mathcal{F}(Ces_1[0, 1], Ces_\infty[0, 1]) \subset (Ces_1[0, 1], Ces_\infty[0, 1])_{1-1/p, \infty}, \quad (41)$$

whence  $Ces_p[0, 1] \subset (Ces_1[0, 1], Ces_\infty[0, 1])_{1-1/p, \infty}$ . But in view of Lemma 3 the last imbedding does not hold, and the proof is complete.  $\square$

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